# Disjoint Cycles in Ordinary Multipartite Tournaments and Round-Robin Tournaments

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**Abstract** In 2010, N. Lichiardopol conjectured for $q \geq 3$ and $k \geq 1$ that any tournament with minimum out-degree at least $(q-1)k-1$ contains $k$ disjoint cycles of length $q$, which has been established for tournaments. In this paper, we demonstrate that the conjecture holds for ordinary multipartite tournaments when $q = 3$, and for round-robin tournaments when $q \geq 3$. Moreover, we point out several flaws found in the proof for tournaments when $q = 4$.

**Keywords:** Lichiardopol conjecture; Minimum out-degree; Disjoint cycles; Ordinary multipartite tournaments; Round-robin tournaments

## Contents

## 1 Introduction

This paper mainly presents a study of digraphs, and we refer the readers to the book [2] for standard definitions related to digraphs. For a digraph $D = (V, A)$, if there is an arc from $x$ to $y$, we write $x \to y$, and call $y$ an *out-neighbor* of $x$, $x$ an *in-neighbor* of $y$. For a vertex $x$ of $D$, we let $d_D^+(x)$ denote the *out-degree* of $x$, which is the number of out-neighbors of $x$. Similarly, we let $d_D^-(x)$ denote the *in-degree* of $x$, let $\delta^+(D)$ denote the minimum out-degree of $V$. For $U \subseteq V$, we let $d_U^+(x)$ denote the number of out-neighbors of $x$ in $U$, similarly $d_U^-(x)$ denote the number of in-neighbors of $x$ in $U$. For two vertex sets $A$ and $B$, we let $d^+(A, B)$ denote the number of arcs from $A$ to $B$. We say there exists a $k$-*matching* from $A$ to $B$ if there exist $k$ arcs from $A$ to $B$ which have no common endpoints. A *tournament* is a digraph such that for each pair of vertices $x$ and $y$, there exists exactly one arc between $x$ and $y$. There are many interesting results about tournaments, we refer interested readers to Chapter 2 of [2]. A digraph $D = (V, A)$ is called a *multipartite tournament* if $V$ can be partitioned into sets $V_1, V_2, ..., V_m$, such that for any $i < j$ and any $v_1 \in V_i, v_2 \in V_j$, there exists exactly one arc between $v_1$ and $v_2$, and for any $v_1, v_2 \in V_i$, there exists no arc between $v_1$ and $v_2$. Moreover, if for any $i < j$, the arcs from $V_i$ to $V_j$ have a common orientation, then the multipartite tournament is called an *ordinary multipartite tournament*. Some properties of ordinary multipartite tournaments are presented by the work of Bang-Jensen et al [3]. An ordinary multipartite tournament is also called a *uniform multipartite tournament* [11]. Note that a tournament is always an ordinary multipartite tournament, if we partition its vertex set into sets each of which contains exactly one vertex.

In this work, we also consider the problem of multi-digraph. The concept of a multi-digraph is a generalization of the concept of a digraph. In a multi-digraph there might be multi-arcs from vertex $x$ to $y$. But multi-digraphs with loops are not addressed in this work. In this paper we define a *round-robin tournament* to be a multi-digraph such that there are exactly two arcs between each pair of vertices. To our best knowledge, round-robin tournaments were first studied in [8]. In this master's thesis round-robin tournaments were called double-arc tournaments. We call them round-robin tournaments because they can model round-robin tournaments in real world.

Problems related to disjoint cycles in digraphs have always been an area of focus. J. C. Bermond and C. Thomassen gave the following conjecture in 1981:

**Conjecture 1.1.** *[4] For any digraph $D$, if $\delta^+(D) \geq 2k-1$, then $D$ contains $k$ disjoint cycles.*

The conjecture is trivial for $k = 1$ and it was proved for $k = 2$ in [10] and for $k = 3$ in [7].

J. Bang-Jensen, S. Bessy and S. Thomassé had a great contribution on this conjecture. In [1] they proved the following theorem:

**Theorem 1.2.** *For $k \geq 1$, every tournament $T$ with $\delta^+(T) \geq 2k-1$ has $k$ disjoint cycles, each of which has length* 3*.*

For convenience, we call a cycle with length $q$ an *$q$-cycle*. Note that if a tournament has an $q$-cycle, then we can find a 3-cycle whose vertex set is a subset of the vertex set of the $q$-cycle. This can be easily proved by induction on $q$. Thus whenever a tournament $T$ contains $k$ disjoint cycles, it contains $k$ disjoint 3-cycles.

Tending to generalize Theorem 1.2 in another dimension, N. Lichiardopol raised another conjecture in 2010:

**Theorem 1.3.** *[6] For $k \geq 1$ and $q \geq 3$, every tournament $T$ with $\delta^+(T) \geq (q-1)k-1$ has $k$ disjoint $q$-cycles.*

When $q = 3$ this conjecture is exactly Theorem 1.2. The case $q = 4$ was proved in the master's thesis of S.Zhu [13]. F. Ma, D. B. West and J. Yan proved this conjecture for $q \geq 5$ in [9].

In this paper we firstly generalize Theorem 1.2 to ordinary multipartite tournament case:

**Theorem 1.4.** *For $k \geq 1$, every ordinary multipartite tournament $T$ with $\delta^+(T) \geq 2k-1$ has $k$ disjoint* 3*-cycles.*

Theorem 1.4 will be proved in Section 2.

In fact, at first we wanted to generalize Theorem 1.3 to ordinary multipartite tournament case, but we found a counterexample when $q = 4$. Hence, we extend Theorem 1.3 to another case, namely the round-robin tournament:

**Theorem 1.5.** *For $k \geq 1$ and $q \geq 3$, every round-robin tournament $T$ with $\delta^+(T) \geq 2(q-1)k-2$ has $k$ disjoint $q$-cycles.*

Theorem 1.5 will be proved in Section 3.

Furthermore, we find that although the final result in [13] is correct but with flaws in the proof. So we list all the flaws in [13] and then present a proof with better completeness in Section 4.

# 2 Disjoint 3-cycles in Ordinary Multipartite Tournaments

## 2.1 Preparation

In order to prove Theorem 1.4,we prove a slightly stronger theorem:

**Theorem 2.1.** *Let $k$ be a positive integer with $k \geq 1$. Suppose $T$ is an ordinary multipartite tournament with $\delta^+(T) \geq 2k-1$. For any $k-1$ disjoint 3-cycles $\mathcal{F} = \{C_1, C_2, \dots, C_{k-1}\}$ in $T$, let $W = V(C_1) \cup V(C_2) \cup \dots V(C_{k-1})$, $U = V(T) \backslash W$, there exist $k$ disjoint 3-cycles whose vertex set intersects $U$ on at most 4 vertices.*

It deserves to be noted that Theorem 1.4 can be directly deduced from Theorem 2.1 by induction on $k$.

We still denote $V(C_i)$ by $C_i$ when it causes no defusion.

First of all, some lemmas frequently used in the proofs are listed below.

**Lemma 2.2.** *In an ordinary multipartite tournament, if $x \to y, y \to z$, then there is an arc between $x$ and $z$.*

*Proof.* Since $x \to y, y \to z$, $x$ and $z$ belong to different parts, thus there is an arc between $x$ and $z$. □

**Lemma 2.3.** *For every acyclic multipartite tournament with $n$ vertices, there is an ordering of the vertices $v_1, v_2, \dots, v_n$, such that for any $i < j$, there is no arc from $v_i$ to $v_j$.*

We can prove this lemma by induction on $n$.

Note that "no arc from $v_i$ to $v_j$" means that either there exists no arc between $v_i$ and $v_j$, or there exists an arc $v_j \to v_i$.

**Lemma 2.4.** *If a multipartite tournament has a cycle, then it has a 3-cycle.*

*Proof.* Suppose the cycle has length $k$. We prove it by induction on $k$.

When $k = 3$, this is obvious.

For $k \geq 4$, let the cycle be $v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow ... \rightarrow v_k \rightarrow v_1$. By Lemma 2.2, there is an arc between $v_1$ and $v_3$. If $v_3 \rightarrow v_1$, we get a 3-cycle $v_1 \rightarrow v_2 \rightarrow v_3$. If $v_1 \rightarrow v_3$, we get a $(k-1)$-cycle $v_1 \rightarrow v_3 \rightarrow ... \rightarrow v_k \rightarrow v_1$. By induction hypothesis we can get the result. □

**Lemma 2.5.** *[5] (See also Theorem 3.1.16 of [12]) Suppose $A$ and $B$ are two disjoint set of vertices in a tournament, and there is no $k$-matching from $A$ to $B$. Then there exists a subset $C$ of $A \cup B$ containing at most $k-1$ vertices, such that the endpoints of all arcs from $A$ to $B$ belong to $C$.*

We will prove Theorem 2.1 by induction on $k$.

When $k = 1$, obviously $T$ has a cycle. By Lemma 2.4 it has a 3-cycle. Thus Theorem 2.1 holds.

For $k \geq 2$, we argue by contradiction. Suppose there exist $k-1$ disjoint 3-cycles $\mathcal{F} = \{C_1, C_2, \ldots, C_{k-1}\}$ in $T$, but do not exist $k$ disjoint 3-cycles which fit the theorem. We name it "the ultimate assumption".

Recall that $W = V(C_1) \cup V(C_2) \cup \ldots V(C_{k-1})$, $U = V(T) \backslash W$. Note that the sub-multipartite tournament of $T$ induced by $U$ is an acyclic ordinary multipartite tournament. Otherwise this sub-multipartite tournament contains a 3-cycle, which contradicts the ultimate assumption.

Moreover, here are two important definitions we need to present:

**Definition 2.6.** *For $i$ 3-cycles in $\mathcal{F}$, $i \in \{1, 2\}$, we say that they can be* extended *if and only if there exist $i+1$ disjoint 3-cycles whose vertices belong to the initial 3-cycles and $U$, and intersect $U$ on at most four vertices.*

Note that once there exists 1 or 2 cycles in $\mathcal{F}$ that can be extended, the ultimate assumption would be violated.

**Definition 2.7.** *For arc $xy, x, y \in W$, and vertices $z, z \in U$, $z$ is a* breaker *of $xy$ if and only if $x \rightarrow y \rightarrow z \rightarrow x$ forms a 3-cycle.*

Notation: Below we denote the 3-cycle $x \rightarrow y \rightarrow z \rightarrow x$ by $(xyzx)$.

## 2.2 Several Prepositive Claims

**Claim 1.** *For every $C_i \in \mathcal{F}$, $C_i$ has at most two arcs with breakers, and every arc has at most three breakers. Thus $C_i$ has at most six breakers.*

*Proof.* Let $C_i = (xyzx)$.

(1) Suppose for contradiction that every arc of $C_i$ has a breaker. Let $v_{xy}$, $v_{yz}$, $v_{zx}$ be breakers of $xy$, $yz$ and $zx$ respectively. Since $v_{zx} \to z \to v_{yz}$, there exists an arc between $v_{zx}$ and $v_{yz}$. Furthermore, we have $v_{zx} \to v_{yz}$. Otherwise, 3-cycles $xyv_{xy}, zv_{yz}v_{zx}$ can extend $C_i$, which violates the ultimate assumption. Symmetrically, we have $v_{yz} \to v_{xy}$ and $v_{xy} \to v_{zx}$, which forms a 3-cycle in $U$. This contradicts the fact that $U$ is acyclic;

(2) For the sake of contradiction, we assume that the arc $xy$ has four breakers, which were named $v_1, v_2, v_3, v_4$ respectively.

Consider the ordinary multipartite tournament $T' = T - \{x, y\}$. We have $\delta^+(T') \geq 2(k-1)-1$, and $T'$ has $k-2$ disjoint 3-cycles, i.e. $\mathcal{F}\backslash\{C_i\}$. Applying induction on $k$, we know that there is a collection $\mathcal{F}'$ in $T'$, which contains $k-1$ disjoint 3-cycles, and $\mathcal{F}'$ intersects $U \cup \{z\}$ on at most four vertices. Since $v_1, v_2, v_3, v_4, z \in U \cup \{z\}$, at least one of those five vertices is not included in $\mathcal{F}'$. In the following content, the selection of vertex is presented.

(2.1) Suppose that $z$ is not included in $\mathcal{F}'$. Thus, the collection $\mathcal{F}' \cup C_i$ contains $k$ disjoint 3-cycles, and its vertices intersect $U$ on at most four vertices. This contradicts the ultimate assumption.

(2.2) Suppose that $z$ is included in $\mathcal{F}'$. Without loss of generality, $v_1$ is not included in $\mathcal{F}'$. Therefore $\mathcal{F}' \cup \{(xv_1yx)\}$ contains $k$ disjoint 3-cycles, and its vertex set intersects $U$ on at most four vertices. This contradicts the ultimate assumption. $\square$

Thanks to Lemma 2.3, $U$ can be divided into two subsets $U_1$ and $U_2$, such that there is no arc from $U_1$ to $U_2$. Also the size of $U_1$ can be an arbitrary integer between 0 and $|U|$.

**Claim 2.** *For arbitrarily chosen partition $U_1, U_2$ of $U$ such that there is no arc from $U_1$ to $U_2$, and for all $i \in [1, k-1]$, the following three properties are true:*

*(1) If $d^+(U_1, C_i) \geq 4$, then there exists a 2-matching from $U_1$ to $C_i$.*

*(2) If $d^+(U_1, C_i) \geq 8$, then there exist a 3-matching from $U_1$ to $C_i$, otherwise $C_i$ can be represented as $(xyzx)$ such that $yz$ has three breakers in $U_1$, $xy$ has two breakers in $U_1$, $d^-_{U_1}(x) \geq 5$, $d^+_{U_2}(x) = 0$, $d^+_{U_2}(y) \leq 1$, $d^+_{U_2}(z) \leq 1$.*

*(3) If $d^+(U_1, C_i) \geq 8$, then there does not exist a 2-matching from $C_i$ to $U_2$, and $d^+(C_i, U_2) \leq 3$.*

By reversing each arc in $T$ and using Claim 2, we can easily obtain the following results:

**The reversing claim.** *(1) If $d^+(C_i, U_2) \geq 4$, then there exists a 2-matching from $C_i$ to $U_2$;*

*(2) If $d^+(C_i, U_2) \geq 8$, then there exist a 3-matching from $C_i$ to $U_2$, otherwise $C_i$ can be represented as $(x'y'z'x')$ such that $x'y'$ has three breakers in $U_2$, $y'z'$ has two breakers in $U_2$, $d^+_{U_2}(z') \geq 5$, $d^-_{U_1}(z') = 0$, $d^-_{U_1}(x') \leq 1$, $d^-_{U_1}(y') \leq 1$;*

*(3) If $d^+(C_i, U_2) \geq 8$, then there does not exist a 2-matching from $U_1$ to $C_i$, and $d^+(U_1, C_i) \leq 3$.*

Now we start to prove Claim 2.

*Proof.* (1) We prove it by contradiction. Assume that there exists no 2-matching from $U_1$ to $C_i$. According to Lemma 2.5, and $E(U_1, C_i) \geq 4$, there exists $x \in C_i$ such that $x$ belongs to all arcs from $U_1$ to $C_i$. Assume in-neighbors of $x$ on $U_1$ are $u_1, u_2, u_3, u_4$. Suppose $x \to y$ in $C_i$, we find that for any $j \in [1, 4], u_j \to x \to y$. $u_j$ and $y$ must therefore be adjacent, and furthermore $y \to u_j$. Hence $u_j$ is a breaker of $xy$, which contradicts Claim 1.

(2) Suppose that there exists no 3-matching from $U_1$ to $C_i$. By Lemma 2.5, there exist $x, y \in U_1 \cup C_i$ such that all arcs from $U_1$ to $C_i$ contain $x$ or $y$. Since $E(U_1, C_i) \geq 8$, $x$ and $y$ can not both be in $U_1$. Without loss of generality, assume that $x \in U_1$ and $y \in C_i$. Now suppose $y \to z$ in $C_i$. Since $E(U_1, C_i) \geq 8$, there exist four vertices(different from $x$) $u_1, u_2, u_3, u_4 \in U_1$ such that for any $j \in [1, 4], u_j \to y \to z$. Hence $u_j$ and $z$ are adjacent. Moreover, $z \to u_j$. Therefore, $u_1, u_2, u_3, u_4$ are $yz$ breakers, which contradicts Claim 1. Thus $x, y \in C_i$.

Without loss of generality, let $C_i = (xyzx)$. Assume $d^-_{U_1}(y) \leq 2$. Since $E(U_1, C_i) \geq 8$, and all arcs from $U_1$ to $C_i$ contain $x$ or $y$, we have $d^-_{U_1}(x) \geq 6$. Thus, there exist four vertices in $U_1$ that have arcs to $x$ but have no arc to $y$. Proceeding with an argument similar to the above, we know that these four vertices are breakers of $yz$, which poses a contradiction. Thus $d^-_{U_1}(y) \geq 3$.

Assume $d^-_{U_1}(y) \geq 4$. There exist four vertices in $U_1$ that have arcs to $y$. Proceeding with an argument similar to the above, we know these four vertices are breakers of $yz$, which poses a contradiction. Thus $d^-_{U_1}(y) \leq 3$.

Now we have $d^-_{U_1}(y) = 3$. Since $E(U_1, C_i) \geq 8$, and all arcs from $U_1$ to $C_i$ contain $x$ or $y$, we have $d^-_{U_1}(x) \geq 5$. Hence there exist at least two vertices in

$U_1$ which have arcs to $x$ but have no arc to $y$. This means $xy$ has at least two breakers in $U_1$. Similarly, we can prove $yz$ has three breakers in $U_1$. Next, let one $xy$ breaker be $x_1$, one $yz$ breaker be $y_1$.

Assume there exists $x' \in U_2$ such that $x \to x'$. We have $x_1 \to x \to x'$, which means $x' \to x_1$. Therefore, $(xx'x_1x)$ and $(yzy_1y)$ extend $C_i$, which contradicts the ultimate assumption. Thus $d^+_{U_2}(x) = 0$.

Assume there exist two vertices $y_a, y_b \in U_2$ which are out-neighbors of $y$. Then we can find that $x \to y \to y_a, y_b$, hence $y_a, y_b$ are both adjacent to $x$. Since $x$ has no out-neighbor in $U_2$, we have $y_a, y_b \to x$. $xy$ has two more breakers, which contradicts Claim 1. Thus $d^+_{U_2}(y) \leq 1$.

Assume there exist two vertices $z_a, z_b \in U_2$ which are out-neighbors of $z$. For the same reason, these two vertices are adjacent to $y$. Because we already have $d^+_{U_2}(y) \leq 1$, then either $z_a$ or $z_b$ is an in-neighbor of $y$. Then $yz$ has one more breakers, which contradicts Claim 1. Thus $d^+_{U_2}(z) \leq 1$.

(3) We prove it by contradiction. Suppose there exists a 2-matching from $C_i$ to $U_2$.

(3.1) If there exists a 3-matching from $U_1$ to $C_i$, then $C_i$ can be extended by two 3-cycles which have the form $U_1 \to C_i \to U_2 \to U_1$;

(3.2) If there does not exist a 3-matching from $U_1$ to $C_i$, then suppose that $C_i = (xyzx)$, $d^+_{U_2}(x) = 0$, and the 2-matching from $C_i$ to $U_2$ is $\{yy', zz'\}$. At this time, $y \to z'$. Otherwise $z'$ will be the fourth breaker of $yz$. Thus, we have $x \to y \to z', x \to y \to y'$. Furthermore, since $d^+_{U_2}(x) = 0$, we have $y', z' \to x$. Hence $y', z'$ are two more breakers of the $xy$, which poses a contradiction.

Consequently, there is no 2-matching from $C_i$ to $U_2$. And by the reversing claim (1) we immediately have $d^+(C_i, U_2) \leq 3$. □

**Claim 3.** *For any $C_i, C_j \in \mathcal{F}$, it is impossible that there exist 3-matchings from $U_1$ to $C_i$, $C_i$ to $C_j$, and $C_j$ to $U_2$.*

*Proof.* Assume that there exist 3-matchings from $U_1$ to $C_i$, $C_i$ to $C_j$, and $C_j$ to $U_2$.

Suppose that $C_i = (xyzx)$ and $C_j = (x'y'z'x')$. Three 3-matchings are $\{x_1x, y_1y, z_1z\}, \{xx', yy', zz'\}$, and $\{x'x_2, y'y_2, z'z_2\}$ respectively. Since $x \to x' \to x_2$, $x$ and $x_2$ are adjacent. For the same reason, $y$ and $y_2$, $z$ and $z_2$ are adjacent. Next, we say $w$ is "positively adjacent" if $w \to w_2$, "negatively adjacent" if $w_2 \to w$, where $k \in \{x, y, z\}$.

(1) All $x, y, z$ are negatively adjacent. At this time, $C_i, C_j$ can be extended by $(xx'x_2x), (yy'y_2y), (zz'z_2z)$;

(2) Exactly one of $x, y, z$ is positively adjacent. Without loss of generality, suppose this vertex is $x$. At this time $x_1 \to x \to x_2$, hence $x_2 \to x_1$. $C_i, C_j$ can be extended by $(xx_2x_1x), (yy'y_2y), (zz'z_2z)$;

(3) At least two of $x, y, z$ are positively adjacent. Without loss of generality, suppose two of them are $x$ and $y$. Now $C_i$ can be extended by $(xx_2x_1x), (yy_2y_1y)$. $\square$

**Claim 4.** *For any $C_i, C_j \in \mathcal{F}$, it is impossible that $d^+(U_1, C_i) \geq 8$, $d^+(C_i, C_j) \geq 7$, and $d^+(C_j, U_2) \geq 8$.*

*Proof.* Assume $C_i$ and $C_j$, whose vertices are $\{x, y, z\}$ and $\{x', y', z'\}$ respectively, satisfy $d^+(U_1, C_i) \geq 8, d^+(C_i, C_j) \geq 7$, and $d^+(C_j, U_2) \geq 8$. Since $d^+(C_i, C_j) \geq 7$, we see that there is a 3-matching from $C_i$ to $C_j$.

(1) Assume there exists no 3-matching from both $U_1$ to $C_i$, and $C_j$ to $U_2$. Suppose that $C_i = (xyzx)$ and $C_j = (x'y'z'x')$. According to Claim 2 and the reversing claim, without loss of generality, we assume that $yz$ has three breakers in $U_1$, $xy$ has at least two breakers in $U_1$, $x'y'$ has three breakers in $U_2$, and $y'z'$ has at least two breakers in $U_2$. Also $d^+_{U_2}(x) = d^-_{U_1}(z') = 0$, $d^+_{U_2}(z) \leq 1$, $d^-_{U_1}(x') \leq 1$. Now suppose $x_1 \in U_1$ is an $xy$ breaker but not an in-neighbor of $x'$, $z_2 \in U_2$ is a $y'z'$ breaker but not an out-neighbor of $z$. And we denote $y_1, y_2$ as breakers of $yz$ and $x'y'$ respectively, which are different from $x_1$ and $z_2$.

Firstly, there is no arc from $x$ to $z'$. Otherwise $C_i, C_j$ can be extended by $(z_2xz'z_2)$, $(yzy_1y)$, $(y'y_2x'y')$. Secondly, there is no arc from $x$ to $x'$. Otherwise, as we already know $x_1 \to x \to x'$, we have $x' \to x_1$. Thus $C_i, C_j$ can be extended by $(x'x_1xx')$, $(z_2y'z'z_2)$, $(yzy_1y)$. Thirdly, there is no arc from $z$ to $z'$. Otherwise, we have $z \to z' \to z_2$, thus $z_2 \to z$. $C_i, C_j$ can be extended by $(z'z_2zz')$, $(x'y'y_2x')$, $(x_1xyz_1)$.

However, $d^+(C_i, C_j) \geq 7$, there must exist an arc from $x$ to $C_j$, we get a contradiction.

(2) Without loss of generality, assume that there exists no 3-matching from $U_1$ to $C_i$, but one from $C_j$ to $U_2$. We let $C_i = (xyzx)$, and denote 3-matching of $C_i$ to $C_j$, $C_j$ to $U_2$ by $\{xx', yy', zz'\}, \{x'x_2, y'y_2, z'z_2\}$ respectively. According to Claim 2, $yz$ has three breakers in $U_1$, $xy$ has two breakers in $U_1$. Specifically, we denote one of the $yz$ breakers as $y_1$. Besides, we know that $d^+_{U_2}(x) = 0$,

$d^+_{U_2}(y) \leq 1$, and $d^+_{U_2}(z) \leq 1$. Next, we will construct the contradiction step by step:

Firstly, we claim that $z \to z_2$. Otherwise, because $z \to z' \to z_2$, we have $z_2 \to z$. Now there are two disjoint 3-cycles, i.e. $(z_2 z z' z_2), (x x' x_2)$, because $y_1 \to y \to y'$, $y_1$ and $y'$ are adjacent.

(a) If $y' \to y_1$, then there exists a 3-cycle $(y y' y_1 y)$, which can extend $C_i, C_j$ with $(z_2 z z' z_2), (x x' x_2)$;

(b) If $y_1 \to y'$, then we have $y_1 \to y' \to y_2$, which means $y_1$ and $y_2$ are adjacent. Furthermore, $y_2 \to y_1$. Thus, there exists a 3-cycle $(y_1 y' y_2 y_1)$, which can extend $C_i, C_j$ with $(z_2 z z' z_2), (x x' x_2)$.

Secondly, we claim that $y \to z_2$. Otherwise, we have $z_2 \to y$, and $z_2$ becomes the fourth breaker of $yz$;

Thirdly, we claim that $\{x_2, y_2\} \to \{y, z\}$. Note that $z_2$ is the only out-neighbor of $y, z$ in $U_2$. Since $y \to y' \to y_2$, we have $y_2 \to y$. Since $y_2 \to y \to z$, we have $y_2 \to z$. Since $x_2 \to x \to y$, we have $x_2 \to y$. Since $x_2 \to y \to z$, we have $x_2 \to z$;

Fourthly we claim that the sub-multipartite tournament of $T$ induced by $\{y', y_2, z, z'\}$ is acyclic. Otherwise, the cycle inside can extend $C_i, C_j$ with $(y_1 y z_2), (x x' x_2)$. Moreover, we have $C_j = (x' y' z' x')$. Otherwise, we have $z' \to y'$, implying that $(y' y_2 z z')$ is a cycle, a contradiction. Also we have $y' \to z$. Otherwise since $y' \to y_2 \to z$, $z \to y'$, $(y' y_2 z y')$ is a cycle, which poses a contradiction;

Fifthly, we claim that $x \to y'$. Otherwise, since $x \to x' \to y$, we have $y' \to x$. Besides, we have $y' \to z$ and $d^+(C_i, C_j) \geq 7$. Hence $x \to z'$, and $z \to x'$. Therefore, the 3-cycles $(x z' z_2 x), (x' x_2 z x'), (y y' y_2 y)$ extend $C_i, C_j$;

Sixthly, we claim that $z' \to x_2$. Otherwise, since $z' \to x' \to x_2$, $x_2 \to z'$. So we find that the 3-cycles $(x_2 z' x' x_2), (x y' y_2), (z y_1 y z)$ extend $C_i, C_j$.

From the above analysis, we know that $(z z' x_2 z), (y' y_2 x y'), (z_2 y_1 y z_2)$ extend $C_i, C_j$, which contradicts the ultimate assumption. $\square$

## 2.3 Analysis on range of $k$

Here, we want to prove that $k \leq 6$. Let $|T| = n$. Then $\frac{n(n-1)}{2} \geq (2k-1)n$, so $|T| \geq 4k - 1$. Thus $|U| = |T| - |W| \geq (4k-1) - (3k-3) = k + 2$. When $k \geq 3$, we let $|U_1| = 5$. Among all 3-cycles in $\mathcal{F}$, we define $I$ as the set of cycles receiving at least eight arcs from $U_1$, $O$ as the set of cycles sending at least eight arcs to $U_2$. At last we define $P$ as $P = \mathcal{F}\backslash(O \cup I)$. Note that $i, o, p$ below are defined as the cardinality of $I, O, P$ respectively.

Firstly, we obtain lower and upper bounds of arcs leaving $U_1$ and entering $T\backslash U_1$ respectively with the following:

$$5(2k-1) - 10 \leq 15i + 7p + 3o. \tag{2.1}$$

The $5(2k-1)$ on the left-hand side is a lower bound of arcs with heads in $U_1$, and 10 represents the number of arcs with heads and tails both in $U_1$. On the right-hand side, since $|U_1| = 5$ and one cycle has 3 vertices, the number of arcs from $U_1$ to $I$ is at most $15i$. By the definition of $I$ and $P$, the number of arcs from $U_1$ to $P$ is at most $7p$. By the definition of $O$ and the reversing claim (3), the number of arcs from $U_1$ to $O$ is at most $3o$. We need to point out that there is no arc from $U_1$ to $U_2$. Thus we get the right side of (2.1). As we have $p = k - 1 - i - o$, we can simplify (2.1) to the following form:

$$3k + 4o - 8 \leq 8i. \tag{2.2}$$

Secondly, we estimate the upper and lower bound of arcs which leave $P \cup I$ and enter $T\backslash(P \cup I)$:

$$3(p+i)(2k-1) - \frac{1}{2} \cdot 3(p+i)[3(p+i)-1] \leq 9po + 6io + 7p + 3i + [15(p+i) - (10k - 15 - 3o)]. \tag{2.3}$$

The left-hand side bounds the number of arcs which have heads in $P \cup I$ but no tails there from below. On the right-hand side, $9po, 6io, 7p, 3i$ bound the number of arcs which are from $P$ to $O$, from $I$ to $O$(by Claim 4), from $P$ to $U_2$(by the definition of $O$ and $P$), and from $I$ to $U_2$(by Claim 2(3)), respectively. $15(p+i)$ is an upper bound of number of arcs between $P \cup I$ and $U_1$ (regardless of direction). And $10k - 15 - 3o = 5(2k-1) - 10 - 3o$ is a lower bound of arcs leaving $U_1$ and entering $P \cup I$. Thus, $15(p+i) - (10k - 15 - 3o)$ is an upper bound of arcs from $P \cup I$ to $U_1$. Substitute $p = k - 1 - i - o$ and (2.2) into (2.3), we have

$$16o^2 + (52 - 13k)o + 4k^2 - 24k \leq 0. \tag{2.4}$$

Since $o$ is a real number, the discriminant of (2.4)

$$-87k^2 + 184k + 2704 \tag{2.5}$$

is at least 0. Hence we can get $k \leq 6$.

## 2.4 Small Cases of $k$

**Claim 5.** *For every $C \in \mathcal{F}$, if $d^+(U_1, C) \geq 10$, then there exists at least one 3-matching from $U_1$ to $C$.*

*Proof.* We prove this by contradiction. Assume that there exists a 3-cycle $C$ such that $d^+(U_1, C) \geq 10$, and there is no 3-matching from $U_1$ to $C$.

By Lemma 2.5, there exist $x, y \in U_1 \cup C$ such that all arcs from $U_1$ to $C$ are adjacent to at least one of them.

(1) If $x, y \in U_1$, we have at most six arcs from $U_1$ to $C$, which is impossible;

(2) Without loss of generality, we assume $x \in U_1, y \in C$. Since $E(U_1, C) \geq 10$, $y$ has at least seven in-neighbors in $U_1 \backslash \{x\}$. These vertices dominate $y$, and $y$ dominates $z$. Thus, these seven vertices must be adjacent to $z$, and all of them are $z$'s out-neighbors. Hence $yz$ has at least seven breakers;

(3) If $x, y \in C$, then we let $C = (xyzx)$. Now, it is true that $d^-_{U_1}(y) \leq 3$, otherwise $yz$ has four breakers. It is also true that $d^-_{U_1}(x) \leq 6$, otherwise $xy$ has four breakers. Therefore, $d^+(U_1, C) \leq 9$, which contradicts the assumption. Consequently, there exists at least one 3-matching from $U_1$ to $C$. □

**Definition 2.8.** *We say that 3-cycle $C$ has a* 3-cover *if and only if there exists a 3-matching from $U_1$ to $C$, or two 2-matchings from $U_1$ to $C$ such that they cover all the three vertices of $C$.*

**Claim 6.** *If there exists a 3-cover from $U_1$ to $C \in \mathcal{F}$, then there is no 2-matching from $C$ to $U_2$. Moreover, we have $d^+(C, U_2) \leq 3$.*

*Proof.* We argue by contradiction. Assume that a 3-cycle $C = (xyzx)$ has a 3-cover from $U_1$, and a 2-matching to $U_2$ ($\{xx', zz'\}$).

(1) Suppose the 3-cover is formed by a 3-matching. We can then extend $C$ in an obvious manner;

(2) Suppose the 3-cover is formed by 2-matchings, and one of them has tails in $\{x, z\}$. We name this 2-matching $\{x_1x, z_1z\}$. This time we find that $C$ can be extended by $(x_1xx'x_1), (z_1zz'z_1)$;

(3) Suppose the 3-cover is formed by 2-matchings $\{ax, by\}$ and $\{cy, dz\}$. Then there exists a 3-cycle $(axx'a)$.

(3.1) If $z'$ and $b$ are adjacent, then the 4-cycle $(byzz'b)$ contains a 3-cycle, which can extend $C$ together with $(axx'a)$.

(3.2) If $z'$ and $b$ are not adjacent, we have $z' \to y$ because $b \to y$. Then 3-cycles $(yzz'y)$ and $(axx'a)$ extend $C$. $\square$

We should point out that Claim 5 and Claim 6 are correct if we reverse $U_1$ and $U_2$, "receiving" and "sending" in the statement.

We name a 3-cycle $C_i \in \mathcal{F}$ as *2-m*, *3-c*, and *3-m* if and only if there exists a 2-matching, 3-cover, and 3-matching from $U_1$ to $C_i$ respectively.

**Claim 7.** *Suppose $a, b, c$ are arbitrarily chosen vertices in $U_1$ and $Y = V(\mathcal{F}')$, where $\mathcal{F}'$ is a subset of $\mathcal{F}$ containing $p$ 3-cycles. If $d_Y^+(a) \geq 2p, d_Y^+(b) \geq 2p - 1, d_Y^+(c) \geq 2p - 2$, then $\mathcal{F}'$ has a 3-c cycle, or all the cycles in $\mathcal{F}'$ are 2-m.*

*Proof.* We prove this claim by induction on $p$.

When $p = 1$, the claim is true.

When $p \geq 2$, let $k = d^+(\{a, b, c\}, Y)$. We have $k \geq 6p - 3 > 3p$. Hence there exists a cycle $C_i \in \mathcal{F}'$ such that $d^+(\{a, b, c\}, C_i) \geq 4$. So $C_i$ is 2-m.

(1) If $C_i$ is 3-c, the induction is proved;

(2) If $C_i$ is not 3-c, then for any $x \in \{a, b, c\}, d^+(x, C_i) \leq 2$. By applying the induction hypothesis on $\mathcal{F}' \backslash \{C_i\}$, we know the statement is true as well. $\square$

Next we will only consider $k \in \{2, 3, 4, 5, 6\}$.

(1) $k = 2$

When $k = 2$, we have $\delta^+(T) \geq 3$. According to Thomassen's work in [10], there exist two disjoint 3-cycles $C_1'$ and $C_2'$ in $T$. Now $\mathcal{F}$ has exactly one 3-cycle $C_1$.

(1.1) If there exists a cycle $C \in \{C_1', C_2'\}$ such that $V(C) \cap V(C_1) = \phi$, then $C, C_1$ extend $C_1$.

(1.2) If both $C_1'$ and $C_2'$ have common vertices with $C_1$, then $C_1', C_2'$ extend $C_1$.

(2) $k = 3$

When $k = 3$, we have $\delta^+(T) \geq 5$. Suppose that $\{C_i, C_j\}$ are two 3-cycles in $\mathcal{F}$. Let $U_1 = \{v_1, v_2\}$. Then there are at least $2 \cdot 5 - 1 = 9$ arcs from $U_1$ to $W$, at most $2 \cdot 6 - 9 = 3$ arcs from $W$ to $U_1$. There are at least $6 \cdot 5 - \frac{1}{2} \cdot 6 \cdot 5 = 15$ arcs from $W$ to $U$, and at least $15 - 3 = 12$ arcs from $W$ to $U_2$. Therefore, we know that there exists a cycle $C_i \in \mathcal{F}$ such that $d^+(U_1, C_i) \geq 5$. Meanwhile $d^+(U_1, C_j) \geq 3$.

(2.1) $d^+(U_1, C_j) \geq 4$: In this case $C_j$ is a 2-m cycle, and $d^+(C_j, U_2) \leq 7$. As a result, $d^+(C_i, U_2) \geq 5$.

Since $d^+(U_1, C_i) \geq 5$, $C_i$ is 3-c. By Claim 6, there does not exist a 2-matching from $C_i$ to $U_2$. But $d^+(C_i, U_2) \geq 5$, by the reversing claim, hence there exists a 2-matching from $C_i$ to $U_2$, which poses a contradiction.

(2.2) $d^+(U_1, C_j) = 3$: If $C_j$ is a 2-m cycle, we can get a contradiction in the same way as (2.1). Thus $C_j$ is not 2-m. Then either $v_1 \rightarrow C_j \rightarrow v_2$, or $v_2 \rightarrow C_j \rightarrow v_1$. If $d^+(U_1, C_i) = 5$, then $d^+(U_1, C_j) \geq 4$, $C_j$ is 2-m, which poses a contradiction. Now we only need to consider the case $d^+(U_1, C_i) = 6$, $U_1 \rightarrow C_i$. If there exists an arc from $C_j$ to $C_i$, then $d^+(C_i, U_2) \geq 5 \times 3 - 3 \times 3 - 3 = 3$. When $E(C_i, U_2) \geq 4$, we can get a contradiction in the same way as (2.1). Now we can suppose $C_i \rightarrow C_j$, and $d^+(C_i, U_2) = 3$.

Let $C_i = (xyzx)$ and $C_j = (x'y'z'x')$. From the above we know each vertex in $C_j$ has at least 3 out-neighbors in $U_2$. Without loss of generality, suppose $x$ has the largest number of out-neighbors in $U_2$. If $d^+(x, U_2) \leq 2$, there exists an out-neighbor of $x'$ in $U_2$ which is not an out-neighbor of $x$ (denoted by $v_1'$). Now $d^+(y, U_2) \leq 1$, $d^+(z, U_2) \leq 1$. There exists an out-neighbor of $y'$ in $U_2$ different from $v_1'$ which is not an out-neighbor of $y$ (denoted by $v_2'$). There exists an out-neighbor of $z'$ in $U_2$ different from $v_1', v_2'$. Now we have three cycles $(xx'v_1'x)$, $(yy'v_2'y)$ and $(zz'v_3'v_iz)$ ($i = 1$ if $v_1 \rightarrow z$ and $i = 2$ if $v_2 \rightarrow z$) to extend $C_i$ and $C_j$, a contradiction. If $d^+(x, U_2) = 3$, then $d^+(y, U_2) = d^+(z, U_2) = 0$. Let $v_1'$ be one out-neighbor of $x$ in $U_2$. There exists an out-neighbor of $y'$ in $U_2$ different from $v_1'$. There exists an out-neighbor of $z'$ in $U_2$ different from $v_1', v_2'$. Now we have three cycles $(v_ixv_1'v_i)$ ($i = 1$ if $v_1 \rightarrow x$ and $i = 2$ if $v_2 \rightarrow x$), $(yy'v_2'y)$ and $(zz'v_3'z)$ to extend $C_i$ and $C_j$, which poses a contradiction.

In the following proof we assume that $U_1 = \{v_1, v_2, v_3\}$.

(3) $k = 4$

When $k = 4$, we have $\delta^+(T) \geq 7$, and there are three 3-cycles in $\mathcal{F}$, we call them $C_1$, $C_2$ and $C_3$. As $|U_1| = 3$, we know that there are at least $3 \cdot 7 - 3 = 18$ arcs from $U_1$ to $W$, at most $3 \cdot 9 - 18 = 9$ arcs from $W$ to $U_1$. There are at least $9 \cdot 7 - \frac{1}{2} \cdot 9 \cdot 8 = 27$ arcs from $W$ to $U$, and at least $27 - 9 = 18$ arcs from $W$ to $U_2$. Therefore, 3-cycles in $\mathcal{F}$ cannot be 2-m, 2-m, and 3-c respectively. Otherwise, there are at most $7 + 7 + 3 = 17$ arcs from $W$ to $U_2$.

**Claim 8.** *At least two 3-cycles in $\mathcal{F}$ are 3-c.*

*Proof.* Because $d^+(U_1) \geq 7$, there are at least 7 arcs from $u_1$ to $W$. So without loss of generality, we suppose $d^+(v_1, C_1) = 3$.

(3.1) If $d^+(\{v_2, v_3\}, C_1) = 0$, then $d^+(v_2, C_2 \cup C_3) \geq 6$, and $d^+(v_3, C_2 \cup C_3) \geq 5$. Thus, $C_2$ and $C_3$ are 3-c;

(3.2) If $d^+(\{v_2, v_3\}, C_1) \geq 1$, then $C_1$ is a 3-c cycle. Note that there are at least 4,3,2 arcs from $v_1, v_2, v_3$ to $C_2 \cup C_3$ respectively. As Claim 7 goes, there are at least one 3-c cycle in $C_2 \cup C_3$. Consequently, the claim is true.

$\square$

Without loss of generality, let $C_1$ and $C_2$ be 3-c cycles. There does not exist a 2-matching from $U_1$ to $C_3$, which means $d^+(U_1, C_3) \leq 3$. As a result, $d^+(U_1, C_1 \cup C_2) \geq 15$, and thus one of $C_1$ and $C_2$ (let it be $C_1$) is 3-m.

Moreover, we can know there are at least $6 \cdot 7 - \frac{1}{2} \cdot 6 \cdot 5 = 27$ arcs from $C_1 \cup C_2$ to $U \cup C_3$. Among them at most $18 - 15 = 3$ to $U_1$, and $2 \cdot 3 = 6$ to $U_2$. Thus there are at least 18 remaining arcs which must go to $C_3$. Therefore there exists a 3-matching from $C_1$ to $C_3$.

As the last step, we know $d^+(C_3, U_2) \geq 18 - 3 - 3 = 12$, hence there exists a 3-matching from $C_3$ to $U_2$. Consequently, there are 3-matchings from $U_1$ to $C_1$, $C_1$ to $C_3$, and $C_3$ to $U_2$. This contradicts Claim 3.

(4) $k = 5$

When $k = 5$, we have $\delta^+(T) \geq 9$, and there are four 3-cycles in $\mathcal{F}$, we call them $C_1$, $C_2$, $C_3$ and $C_4$. Similar to case (3), we get $d^+(U_1, W) \geq 24$, $d^+(W, U_1) \leq 12$, $d^+(W, U) \geq 42$, and $d^+(W, U_2) \geq 30$. Therefore, 3-cycles in $\mathcal{F}$ cannot all be 2-m cycles. Otherwise, $d^+(W, U_2) \leq 4 \cdot 7 = 28$.

**Claim 9.** *At least two cycles in $\mathcal{F}$ are 3-c cycles.*

*Proof.* As there are at least 9 arcs from $u_1$ to $W$, we suppose $d^+(u_1, C_1) = 3$ without loss of generality.

(4.1) If $d^+(\{v_2, v_3\}, C_1) = 0$, then $d^+(v_2, C_2 \cup C_3 \cup C_4) \geq 8$, and $d^+(v_3, C_2 \cup C_3 \cup C_4) \geq 7$. Without loss of generality, let $v_2 \to C_2 \cup C_3$. As we know $d^+(v_3, C_2 \cup C_3) \geq 4$, then $u_3$ is adjacent to both cycles. Thus, $C_2, C_3$ are 3-c cycles.

(4.2) If $d^+(v_2, v_3, C_1) \geq 1$, then $C_1$ is a 3-c cycle. There are at least 6,5,4 arcs from $v_1, v_2, v_3$ to $C_2 \cup C_3 \cup C_4$ respectively. By Claim 7, there are at least one 3-c cycle in $C_2 \cup C_3 \cup C_4$. Consequently, the claim is true. $\square$

Without loss of generality, let $C_1$ and $C_2$ be 3-c cycles.Then there exist at least 3,2,1 arcs from $v_1, v_2, v_3$ to $C_3 \cup C_4$ respectively. Therefore one of these two cycles, let it be $C_3$, is 2-m. Then $C_4$ is not 2-m, which means that $d^+(U_1, C_4) \leq 3$. As a result, $d^+(U_1, C_1 \cup C_2 \cup C_3) \geq 21$. We observe that for any $C_i, i = 1, 2, 3$, if $C_i$ is not a 3-m cycle, $d^+(U_1, C_i) \leq 6$. Hence at least one of $C_1, C_2, C_3$ is 3-m. Let this cycle be $C_1$.

Moreover, we notice that $d^+(C_1 \cup C_2 \cup C_3, U_1) \leq 3 \cdot 9 - 21 = 6$, $d^+(C_1 \cup C_2 \cup C_3, U_2) \leq 3{+}3{+}7 = 13$, and $d^+(C_1 \cup C_2 \cup C_3, T \backslash (C_1 \cup C_2 \cup C_3)) \geq 9 \cdot 9 - \frac{1}{2} \cdot 9 \cdot 8 = 45$. Thus $d^+(C_1 \cup C_2 \cup C_3, C_4) \geq 45 - 13 - 6 = 26$, which means that $d^+(C_1, C_4) \geq 8$, and there exists a 3-matching from $C_1$ and $C_4$.

As the last step we know that $d^+(C_4, U_2) \geq 30 - 13 = 17$. Hence there exists a 3-matching from $C_4$ to $U_2$. So far, we have found 3-matchings from $U_1$ to $C_1$, $C_1$ to $C_4$, and $C_4$ to $U_2$. This contradicts Claim 3.

(5) $k = 6$

When $k = 6$, we have $\delta^+(T) \geq 11$, and there are five 3-cycles in $\mathcal{F}$, we call them $C_1$, $C_2$, $C_3$, $C_4$ and $C_5$. Similar to case (3), we can get $d^+(U_1, W) \geq 30$, $d^+(W, U_1) \leq 15$, $d^+(W, U) \geq 60$, and $d^+(W, U_2) \geq 45$. Therefore, 3-cycles in $\mathcal{F}$ cannot all be 2-m cycles. Otherwise, $d^+(W, U_2) \leq 5 \cdot 7 = 35$.

**Claim 10.** *There are either at least three 3-c cycles, or two 3-c and two 2-m cycles in $\mathcal{F}$.*

*Proof.* As there are at least 11 arcs from $u_1$ to $W$, we suppose $d^+(v_1, C_1) = 3$ without loss of generality.

(5.1) If $d^+(\{v_2, v_3\}, C_1) = 0$, then $d^+(v_2, C_2 \cup C_3 \cup C_4 \cup C_5) \geq 10$, and $d^+(v_3, C_2 \cup C_3 \cup C_4 \cup C_5) \geq 9$. Without loss of generality, let $v_2 \to C_2 \cup C_3$. We know that $d^+(v_3, C_2 \cup C_3) \geq 3$.

(5.1.1) If $v_3$ is connected to both $C_2$ and $C_3$, then $C_2, C_3$ are both 3-c cycles.

(5.1.2) If $v_3$ has no arc to one of these two cycles (let it be $U_2$), then $v_3 \to C_3 \cup C_4 \cup C_5$, and $d^+(v_2, C_4 \cup C_5) \geq 4$. Now, $C_4$ and $C_5$ are 3-c cycles.

(5.2) If $d^+(\{v_2, v_3\}, C_1) \geq 1$, then $C_1$ is a 3-c cycle. Note that there are at least 8,7,6 arcs from $v_1, v_2, v_3$ to $C_2 \cup C_3 \cup C_4 \cup C_5$ respectively. As Claim 7 goes, there are at least one 3-c cycle in $C_2 \cup C_3 \cup C_4 \cup C_5$.

As a result, we know that there exist at least two 3-c cycles in $\mathcal{F}$(let them be $C_1$ and $C_2$). Now there are at least $5, 4, 3$ arcs from $v_1, v_2, v_3$ to $C_3 \cup C_4 \cup C_5$ respectively. Without loss of generality, we assume that $d^+(\{v_1, v_2, v_3\}, C_3) \geq 4$, $C_3$ is then a 2-m cycle.

(5.2.1) If $C_3$ is a 3-c cycle, we get three 3-cycles $C_1$, $C_2$ and $C_3$, and the claim is proven.

(5.2.2) If $C_3$ is not a 3-c cycle, then each vertex of $\{u_1, u_2, u_3\}$ has at least two arcs to $C_3$. This means there are at least 3,2,1 arcs from $v_1, v_2, v_3$ to $C_4 \cup C_5$ respectively. Thus without loss of generality, $C_4$ is 2-m. Hence the claim is also true. $\square$

Now consider the case when $C_1, C_2, C_3$ are 3-c cycles. Now we assume $C_4, C_5$ are not 2-m. Otherwise, it can be dealt with as in the next case. Thus, $d^+(U_1, C_4) \leq 3$, and $d^+(U_1, C_5) \leq 3$. As a result, we have $d^+(U_1, C_1 \cup C_2 \cup C_3) \geq 30 - 3 \cdot 2 = 24$. This result indicates that without loss of generality, $C_1, C_2$ are both 3-m cycles, and $d^+(C_1 \cup C_2 \cup C_3, U_1) \leq 3 \cdot 9 - 24 = 3$.

According to Claim 6, $d^+(C_1 \cup C_2 \cup C_3, U_2) \leq 9$, and $d^+(C_1 \cup C_2 \cup C_3, T \backslash (C_1 \cup C_2 \cup C_3)) \geq 9 \cdot 11 - \frac{1}{2} \cdot 9 \cdot 8 = 63$. Hence $d^+(C_1 \cup C_2 \cup C_3, C_4 \cup C_5) \geq 63 - 9 - 3 = 51$. So one of $C_1$, $C_2$ and $C_3$ (let it be $C_1$), has 3-matchings to both $C_4$ and $C_5$.

As the last step, we have $d^+(C_4 \cup C_5, U_2) \geq 45 - 9 = 36$. Assume that $d^+(C_4, U_2) \geq 18$, then there exists a 3-matching from $C_4$ to $U_2$. Now, there are 3-matchings from $U_1$ to $C_1$, $C_1$ to $C_4$, and $C_5$ to $U_2$. This contradicts Claim 3.

Next, we consider the case where $C_1, C_2$ are 3-c, and $C_3, C_4$ are 2-m. Obviously, $C_5$ is not 2-m and thus $d^+(U_1, C_5) \leq 3$. Therefore, $d^+(U_1, C_1 \cup C_2 \cup C_3 \cup C_4) \geq 27$, $d^+(C_1 \cup C_2 \cup C_3 \cup C_4, U_1) \leq 9$, $d^+(C_1 \cup C_2 \cup C_3 \cup C_4, U_2) \leq 3 \cdot 2 + 7 \cdot 2 = 20$, and $d^+(C_1 \cup C_2 \cup C_3 \cup C_4, T \backslash (C_1 \cup C_2 \cup C_3 \cup C_4)) \geq 12 \cdot 11 - \frac{1}{2} \cdot 12 \cdot 11 = 66$. Consequently, we have $d^+(C_1 \cup C_2 \cup C_3 \cup C_4, C_5) \geq 37$, which is obviously impossible.

Now, when $k = 2, 3, 4, 5, 6$, we can always reach a contradiction. Thus the proof of Theorem 2.1 is completed.

# 3 Disjoint Cycles in Round-Robin Tournaments

In this section we suppose $T$ is a round-robin tournament.

**Definition 3.1.** *If $uv$ and $vu$ are both arcs in $T$, we call $vu$ an opposite arc of $uv$.*

To prove Theorem 1.5, we prove the following theorem in order to "extract" an tournament from an round-robin tournament:

**Theorem 3.2.** *For any positive integer $d \geq 1$, any round-robin tournament $T$ satisfying $\delta^+(T) \geq 2d$ contains a tournament $T'$ satisfying $\delta^+(T') \geq d$.*

*Proof.* We delete the arcs of $T$ step by step until it becomes an empty graph, and reconstruct $T'$.

- For any pair of vertices $u$ and $v$, if there are two arcs from $u$ to $v$, we delete exactly one of them and reach a digraph $T_1'$, we delete both of them and reach a digraph $T_1$.
- If there exists a cycle $C_1$ in $T_1$, we delete arcs in $C_1$ and opposite arcs of $C_1$. If there still exists a cycle $C_2$, we delete arcs in $C_2$ and opposite arcs of $C_2$. Continue this process until there are no cycles. We get a series of edge-disjoint cycles $C_1, C_2, \ldots, C_p$ and finally we reach an acyclic digraph $T_2$.
- If there exists some paths in $T_2$, we find a longest path $P_1$, and delete arcs in $P_1$ and opposite arcs of $P_1$. If there still exists some paths, we find a longest path $P_2$, and delete arcs in $P_2$ and opposite arcs of $P_2$. Continue this process until we reach an empty graph. We get a series of path $P_1, P_2, \ldots, P_q$.

Let $T'$ be $T_1' \cup C_1 \cup C_2 \cup \cdots \cup C_p \cup P_1 \cup P_2 \cup \cdots \cup P_q$. Obviously $T'$ is a tournament. Next we only need to prove $\delta^+(T') \geq d$.

For any vertex $v$, if $v$ is not an endpoint of any path $P_i$, then the out-degree of $v$ in $T$ must be an even number $2k$, and its out-degree in $T'$ must be $k$.

If $v$ is an endpoint of some paths in $\{P_1, \ldots P_q\}$, then suppose $P_i$ is the path which has the smallest subscription. Before deleting arcs in $P_i$, the digraph is acyclic, and $P_i$ is a longest path. Thus after deleting arcs in $P_i$ and opposite arcs of $P_i$, $v$ becomes an isolated vertex. Hence $v$ is not an endpoint of $P_{i+1}, P_{i+2}, \ldots P_q$. This means that $v$ is the endpoint of exactly one path in $\{P_1, \ldots P_q\}$. Consequently, the out-degree of $v$ in $T$ must be an odd number $2k+1$, and its out degree in $T'$ must be $k$ or $k+1$. As $2k+1 \geq 2d$, we have $k \geq d$. Then we complete the proof. $\square$

# 4 List of Flaws

In [13], the author proved Theorem 1.3 when $q = 4$. We restate it as the following theorem:

**Theorem 4.1.** *For any positive integer $k \geq 1$, any tournament with minimum out-degree at least $3k-1$ contains $k$ disjoint cycles of length 4.*

Actually the author proved a slightly stronger argument than Theorem 4.1.

**Theorem 4.2.** *For any positive integer $k \geq 1$, if $T$ is a tournament which has $\delta^+(T) \geq 3k-1$, then for any $k-1$ disjoint 4-cycles $\mathcal{F} = \{C_1, C_2, \ldots, C_{k-1}\}$, let $W = V(C_1) \cup V(C_2) \cup \ldots V(C_{k-1})$ and $U = V(T) \backslash W$, there exist $k$ disjoint 4-cycles whose vertex set intersects $U$ on at most seven vertices.*

We carefully read [13], and found the proofs in [13] are mostly correct. However, there are some typos in [13]. And some proofs are not rigorous. Here we correct those typos and offer rigorous proofs of some claims, so that other readers can understand the result easier.

The author proved Theorem 4.2 by induction on $k$. For $k \geq 2$, the author assumed that there exist $k-1$ disjoint 4-cycles $\mathcal{F} = \{C_1, C_2, \ldots, C_{k-1}\}$, let $W = V(C_1) \cup V(C_2) \cup \ldots V(C_{k-1})$ and $U = V(T) \backslash W$, there do not exist $k$ disjoint 4-cycles whose vertex set intersects $U$ on at most seven vertices.

According to Rédei's Theorem (see Theorem 2.2.4 of [1]), any tournament has a Hamiltonian dipath. We can order the vertices of $U$ as $v_1, v_2, \ldots, v_n$, such that for every $i$, $v_{i+1} \rightarrow v_i$. Obviously the sub-digraph of $T$ induced by $U$ has no 4-cycles, thus there is no arc from $v_i$ to $v_j$ when $j - i \geq 3$.

From Claim 1 to Claim 6 the author supposed that $U_1 = \{v_1, v_2, \ldots, v_6\}$, $S = \{v_7\}$, and $U_2 = U \backslash (U_1 \cup S)$.

**Claim 1.** *For any 4-cycle $C \in \mathcal{F}$, every 3-path of $C$ has at most six breakers.*

**Claim 2.** *Let $C \in \mathcal{F}$. If $d^+(U_1, C) \geq 13$, then there exists a 3-matching from $U_1$ to $C$.*

**Claim 3.** *Let $C \in \mathcal{F}$. If $d^+(C, U_2) \geq 7$, then there exists a 2-matching from $C$ to $U_2$. If $d^+(C, U_2) \geq 13$, then there exists a 3-matching from $C$ to $U_2$.*

**Claim 4.** *Suppose $C_i$ and $C_j$ are two 4-cycles in $\mathcal{F}$. If $d^+(U_1, C_i) \geq 13$ and $d^+(C_j, U_2) \geq 13$, then $d^+(C_i, C_j) \leq 12$.*

**Claim 5.** *Let $S_1$ and $S_2$ be two disjoint vertex sets satisfying $|S_1| \leq 4$ and $|S_2| = 4$. If $d^+(S_1, S_2) \geq 5$, then there exists a 2-matching from $S_1$ to $S_2$. If $d^+(S_1, S_2) \geq 9$, then there exists a 3-matching from $S_1$ to $S_2$.*

**Claim 6.** *Let $C \in \mathcal{F}$. If there exists a 3-matching from $U_1$ to $C$, then there is no 3-matching from $C$ to $U_2$. Conversely, if there exists a 3-matching from $C$ to $U_2$, then there is no 3-matching from $U_1$ to $C$.*

After proving Claim 1 to Claim 6, the author proved $k \leq 7$. After that the author proved that $k = 2, 3, 4, 5, 6$ are all impossible. During the course of the proofs the author supposed that $U_1 = \{v_1, v_2, v_3, v_4\}$, $S = \{v_5, v_6\}$, and $U_2 = U \backslash (U_1 \cup S)$.

We point that the author mainly has four flaws in the proof:

**Flaw 1.** *The proof of Claim 4 is not rigorous.*

In order to offer a rigorous proof of Claim 4, firstly we prove another claim:

**Claim 4.1.** *Suppose $C_i$ and $C_j$ are two 4-cycles in $\mathcal{F}$. If $\{v_{i_1}u_1, v_{i_2}u_2, v_{i_3}u_3\}$ is a 3-matching from $U_1$ to $C_i$ and $\{u'_1v_{j_1}, u'_2v_{j_2}, u'_3v_{j_3}\}$ is a 3-matching from $C_j$ to $U_2$), then there does not exist a 3-matching from $\{u_1, u_2, u_3\}$ to $\{u'_1, u'_2, u'_3\}$.*

*Proof.* Without loss of generality suppose that $i_3 > i_2 > i_1$ and $j_1 > j_2 > j_3$. Assume that there exists a 3-matching from $\{u_1, u_2, u_3\}$ to $\{u'_1, u'_2, u'_3\}$.

(1) If $u_3u'_3$ is in the matching, we extend $C_i$ and $C_j$ by $(v_{i_3}u_3u'_3v_{j_3} \ldots v_{i_3})$, $(v_{i_2}u_2u'_2v_{j_2}v_{i_2})$ and $(v_{i_1}u_1u'_1v_{j_1}v_{i_1})$ (if $u_1u'_1$ is in the matching), or by $(v_{i_3}u_3u'_3v_{j_3} \ldots v_{i_3})$, $(v_{i_2}u_2u'_1v_{j_1}v_{i_2})$ and $(v_{i_1}u_1u'_2v_{j_2}v_{i_1})$ (if $u_1u'_2$ is in the matching);

(2) If $u_3u'_2$ is in the matching, we extend $C_i$ and $C_j$ by $(v_{i_3}u_3u'_2v_{j_2}v_{i_3})$, $(v_{i_2}u_2u'_3v_{j_3}v_{i_2})$, and $(v_{i_1}u_1u'_1v_{j_1}v_{i_1})$ (if $u_1u'_1$ is in the matching), or by $(v_{i_3}u_3u'_2v_{j_2}v_{i_3})$, $(v_{i_2}u_2u'_1v_{j_1}v_{i_2})$, and $(v_{i_1}u_1u'_3v_{j_3}v_{i_1})$ (if $u_1u'_3$ is in the matching);

(3) If $u_3u'_1$ is in the matching, we extend $C_i$ and $C_j$ by $(v_{i_3}u_3u'_1v_{j_1}v_{i_3})$, $(v_{i_2}u_2u'_3v_{j_3}v_{i_2})$, and $(v_{i_1}u_1u'_2v_{j_2}v_{i_1})$ (if $u_1u'_2$ is in the matching), or by $(v_{i_3}u_3u'_1v_{j_1}v_{i_3})$, $(v_{i_2}u_2u'_2v_{j_2}v_{i_2})$, and $(v_{i_1}u_1u'_3v_{j_3}v_{i_1})$ (if $u_1c_3$ is in the matching).

□

Now we are ready to offer the proof of Claim 4.

*Proof.* Let $C_i$ be $(xyztx)$ and $C_j$ be $(x'y'z't'x')$ respectively. Because $d^+(U_1, C_i) \geq 13$ and $d^+(C_j, U_2) \geq 13$, by Claim 2 and Claim 3 there exists one 3-matching from $U_1$ to $C_i$, and one from $C_j$ to $U_2$. Suppose they are $\{v_{i_1}x, v_{i_2}y, v_{i_3}z\}$ and $\{x'v_{j_1}, y'v_{j_2}, z'v_{j_3}\}$ without loss of generality.

Assume that $d^+(C_i, C_j) \geq 13$. As $d^+(\{t\}, C_j) \leq 4$, we have $d^+(\{x, y, z\}, C_j) \geq 9$. We consider three sub-cases:

(1) At least two of $x, y, z$ have 4 arcs to $C_j$. Without loss of generality, we suppose that $d^+(x, C_j) = 4, d^+(y, C_j) = 4$. Since $d^+(\{x, y, z\}, C_j) \geq 9$, we have $d^+(z, C_j) \geq 1$. If $z$ dominates at least one vertex in $\{x', y', z'\}$, then there exists a 3-matching from $\{x, y, z\}$ to $\{x', y', z'\}$. This contradicts Claim 4.1. If $z$ does not dominate anyone of $\{x', y', z'\}$, then $z \to t'$. In this case we can construct an "almost" 3-matching $\{xy', yz', zt'x'\}$ from $\{x, y, z\}$ to $\{x', y', z'\}$. This can cause a contradiction in the same way as the proof of Claim 4.1;

(2) Exactly one of $x, y, z$ has 4 arcs to $C_j$. Without loss of generality we assume that $d^+(x, C_j) = 4, d^+(y, C_j) = 3$, and $d^+(z, C_j) \geq 2$. As a result, $z$ dominates at least one vertex in $\{x', y', z'\}$. Suppose $z \to z'$. $y$ dominates at least one vertex in $\{x', y'\}$. Suppose $y \to y'$. At last, we have $x \to x'$. Hence we have a 3-matching from $C_i$ to $C_j$, which contradicts Claim 4.1;

(3) All of $x, y, z$ have at most 3 arcs to $C_j$. In this case we have $d^+(x, C_j) = d^+(y, C_j) = d^+(z, C_j) = 3$, and thus $d^+(t, C_j) = 4$. Hence, $x$ dominates at least one vertex in $\{x', y', z'\}$. Suppose $x \to x'$. $y$ dominates at least one vertex in $\{y', z'\}$. Suppose $y \to y'$. At last, we have $t \to z'$, which means that there exists an "almost" 3-matching $\{xx', yy', ztz'\}$ from $\{x, y, z\}$ to $\{x', y', z'\}$. This can cause a contradiction in the same way as the proof of Claim 4.1. □

**Flaw 2.** *There are four typos in the proof of Claim 6.*

*(1) In sub-case (1), third paragraph, sixth row, "*$C = (v_3 u_{j_1} \dots u_{j_2} u_{k_2} v_3)$*" should be "*$(v_3 u_{j_1} \dots u_{j_2} u_{k_2} v_3)$*;"*

*(2) In sub-case (2), third paragraph, sixth row, "*$B = (v_3 u_{j_2} \dots u_{j_1} u_{k_1})$*" should be "*$(v_3 u_{j_2} \dots u_{j_1} u_{k_1} v_3)$*;"*

*(3) In sub-case (3), third paragraph, third row, "*$(v_1 u_{j_3} \dots u_7 u_{k_3} v_1)$*" should be "*$(v_1 u_{j_3} \dots u_7 u_{k_1} v_1)$*;"*

*(4) In sub-case (3), third paragraph, fifth row, "*$(v_1 u_{j_3} \dots u_7 u_{k_3} v_1)$*" should be "*$(v_1 u_{j_3} \dots u_7 u_{k_1} v_1)$*."*

**Flaw 3.** *The proof of "*$k = 4$ *is impossible" is not rigorous.*

*Proof.* When $k = 4$, we have $\delta^+(T) \geq 11$, and there are three cycles in $\mathcal{F}$. On account of $d^+(u_1, \mathcal{F}) \geq 10$ and $d^+(u_2, \mathcal{F}) \geq 9$, there exists a cycle $C$ in $\mathcal{F}$ such that $d^+(U_1, C) \geq 7$. Let $C$ be $(xyzt)$. Without loss of generality, we suppose $x \to y$, $y \to z, t$, $z \to t, x$ and $t \to x$.

(1) $d^+(v_1, C) = 4$, $d^+(v_2, C) \geq 3$. Obviously at most one vertex in $C$ is not dominated by $v_2$. There are four sub-cases:

Firstly, $v_2 \to \{x, y, z\}$. We will have $d^+(x, U_2) \geq 2$, $d^+(t, U_2) \geq 1$. There exist arcs from $x$ to $U_2$ $xv_i, xv_j$, $(i < j)$ and an arc from $t$ to $U_2$ $tv_l$. Therefore, $C$ can be extended by: (a) $(v_2ytv_lv_2)$ and $(v_1zxv_jv_1)$ $(j \neq l)$, (b) $(v_2zxv_iv_2)$ and $(v_1ytv_lv_1)$ $(j = l)$.

Secondly, $v_2 \to \{x, y, t\}$. We will have $d^+(x, U_2) \geq 2$, $d^+(t, U_2) \geq 2$. There exist arcs from $x$ to $U_2$ $xv_i, xv_j$, $(i < j)$ and arcs from $t$ to $U_2$ $tv_l, tv_m$, $(l < m)$. Therefore, $C$ can be extended by: (a) $(v_2ytv_lv_2)$ and $(v_1zxv_jv_1)$ $(j \neq l)$, (b) $(v_2xv_j \ldots v_2)$ and $(v_1ytv_mv_1)$ $(j = l)$.

Thirdly, $v_2 \to \{y, z, t\}$. We will have $d^+(x, U_2) \geq 1$, $d^+(t, U_2) \geq 2$. There exists an arc $x$ to $U_2$ $xv_j$ and arcs from $t$ to $U_2$ $tv_l, t, v_m$, $(l < m)$. Therefore, $C$ can be extended by: (a) $(v_2ytv_lv_2)$ and $(v_1zxv_jv_1)$ $(j \neq l)$, (b) $(v_2zxv_jv_2)$ and $(v_1ytv_mv_1)$ $(j = l)$.

Fourthly, $v_2 \to \{x, z, t\}$. We will have $d^+(x, U_2) \geq 2$, $d^+(t, U_2) \geq 2$. There exist arcs from $x$ to $U_2$ $xv_i, x, v_j$, $(i < j)$ and arcs from $t$ to $U_2$ $tv_l, tv_m$, $(l < m)$. Therefore, $C$ can be extended by: (a) $(v_2zxv_iv_2)$ and $(v_1ytv_mv_1)$ $(m \neq i)$, (b) $(v_2zxv_jv_2)$ and $(v_1ytv_mv_1)$ $(m = i)$.

(2) $d^+(v_1, C) \geq 3$, $d^+(v_2, C) = 4$. We can exchange the role of $v_1$ and $v_2$ above and get a proof.

Consequently, we can extend $C$ in all cases, which poses a contradiction.

□

**Flaw 4.** *The proof of "$k = 3$ is impossible" is not rigorous.*

*Proof.* When $k = 3$, we have $\delta^+(T) \geq 8$, and there are two cycles in $\mathcal{F}$. On account of $d^+(u_1, \mathcal{F}) \geq 7$ and $d^+(u_2, \mathcal{F}) \geq 6$, there exists a cycle $C$ in $\mathcal{F}$ such that $d^+(U_1, C) \geq 7$. Hence, at most one vertex in $C$ is not dominated by one vertex in $U_1$. Let $C$ be $(xyzt)$. Without loss of generality, we suppose $x \to y$, $y \to z, t$, $z \to t, x$ and $t \to x$.

It is easy to see that $d^+(x, U_2) \geq 2$ and $d^+(t, U_2) \geq 2$. Let one out-neighbor of $x$ in $U_2$ be $v_i$. Let one out-neighbor of $t$ in $U_2$ different from $v_i$ be $v_j$. Without loss of generality, we suppose $v_1 \to y$ and $v_2 \to z$. Then two 4-cycles $(v_1ytv_jv_1)$ and $(v_2zxv_iv_2)$ can extend $C$, which poses a contradiction.

□

# References


[1] Bang-Jensen, J., Bessy, S., Thomassé, S., Disjoint 3-cycles in tournaments: a proof of the Bermond-Thomassen conjecture for tournaments, *Journal of Graph Theory*, 2014, 75(3): 284–302.

[2] Bang-Jensen, J., Gutin, G., *Digraphs: theory, algorithms and applications, 2nd edition*, Springer-Verlag, London, 2009.

[3] Bang-Jensen, J., Gutin, G., Huang, J., Weakly Halmiltonian-connected ordinary multipartite tournaments, *Discrete Mathematics*, 1995, 138: 63–74.

[4] Bermond, J. C., Thomassen, C., Cycles in digraphs–a survey, *J. Graph Theory*, 1981, 5(1): 1–43.

[5] Kőnig, Dénes, Gráfok és mátrixok, *Matematikai és Fizikai Lapok*, 1931, 38: 116–119.

[6] Lichiardopol, N., Vertex-disjoint directed cycles of prescribed length in tournaments with given minimum out-degree and in-degree, *Discrete Mathematics*, 2010, 310:2567–2570.

[7] Lichiardopol, N., Pór, A., Sereni, J.-S., A step towards the Bermond-Thomassen conjecture about disjoint cycles in digraphs, *Siam J. Discrete Math*, 2009, 23:979–992.

[8] Lin, X., *On some problems of double-arc tournaments*, Master's thesis, Xiamen University, 2011 (in Chinese).

[9] Ma, F., West, D. B., Yan, J., Lichiardopol's conjecture on disjoint cycles in tournaments, Preprint, http://arxiv.org/pdf/1707.02384v2.pdf.

[10] Thomassen, C., Disjoint cycles in digraphs, *Combinatorica*, 1983, 3(3-4):393–396.

[11] Volkmann, L., Multipartite tournaments: A survey, *Discrete Mathematics*, 2007, 307:3097–3129.

[12] West, D. B., *Introduction to graph theory*, Prentice Hall, 2001.

[13] Zhu, S., *On vertex-disjoint cycles in tournaments*, Master's thesis, Shandong University, 2019 (in Chinese).

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